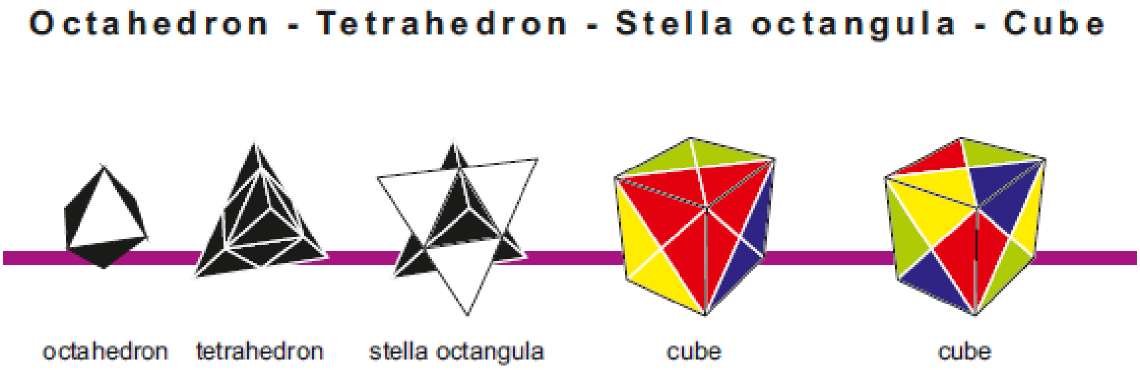
Maths workshop with *From octahedron to cube* exhibit

Description:
The exhibit is a puzzle of 21 wooden pieces linkable by magnets.
The core piece is a black and white octahedron. Its diagonal has the same length as the edge of the cube that represents the solution of the puzzle.
This octahedron is the dual polyhedron of the cube.
The other pieces are: 8 regular and 12 irregular tetrahedrons (namely of type tetr1 and tetr2 respectively)

The activity proposed in the exhibition is simply to build the reproduced solids in the illustrated sequence, starting with the regular octahedron in the centre of the structure:



The only difficulty is to correctly combine the pieces so that the faces of the cube (second option) always present the four colours.

It is a fairly easy activity that can play any type of public in a time appropriate to the rhythms of an exposure.
The magnets hidden in the faces of the different solids make the activity quick and pleasant.
The most interested visitors can discover a bit more mathematics. For example: the opposite faces of the cube have an identical distribution of colours, but in a dual sequence:

However, it is an exhibit that lends itself to being the subject of an Instant Workshop that can be developed in part or in full, depending on the needs and skills of the users.

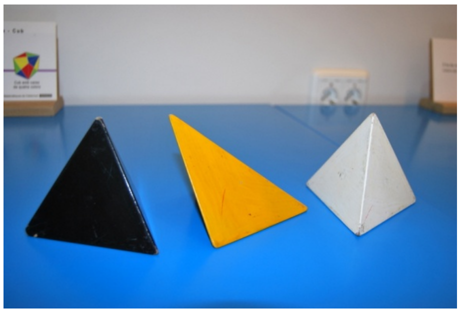
Activity 1

Prove that regular and irregular tetrahedra have the same volume.

It is enough to check that both solids have identical base and equal height.

Question: Do they have the same total area?

Comparing every face of the two tetrahedra, it is easy to prove which one has a greater total area.



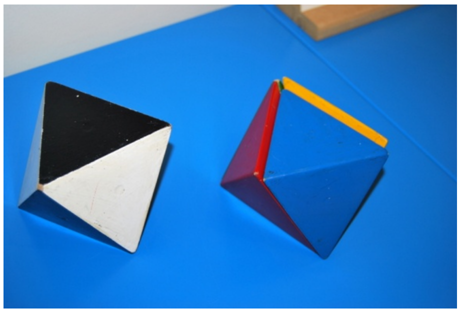
NOTE: it is the same problem as proving that amongst all triangles sharing a common side and equal area, the isosceles has least perimeter.

Activity 2

Calculate the volume of core octahedron in terms of the volume of one regular tetrahedron.

If we join 4 irregular tetrahedra as shown in the figure, it is easy to prove that:

$$Vol_{oct} = 4 \times Vol_{tetr}$$



Question: Will I say that it is the same with regular tetrahedrons? Why?

Activity 3

Build a regular tetrahedron with one regular tetrahedron and three irregular tetrahedra.

It is quite easy to find the solution (see red shape) and to prove that:

$Vol_{mediumtetr} = Vol_{tetr1} + 3 \times Vol_{tetr2} = 4 \times Vol_{tetr}$

Question: Can you compare the volumes of medium tetrahedron and the core octahedron?

Activity 4

Build a regular tetrahedron with the octahedron and 4 regular tetrahedrons.

It is quite easy to find the solution (see black and white body) and to prove that:

$Vol_{bigtetr} = Vol_{oct} + 4\times Vol_{tetr1} =$

$= 4\times Vol_{tetr2} + 4\times Vol_{bigtetr} = 8\times Vol_{tetr}$

Question: What is the volume of the big tetrahedron, in terms of the volume of the octahedron?

Activity 5

Buid an irregular octahedron with 2 regular and 2 irregular tetrahedra.
Thanks to the reflections made in carrying out the previous activities, the observation that the octahedron generated has the same volume and greater area of the regular octahedron is quite obvious (1).
More interesting is continuing to build other octahedra equal to this (2).
Having 8 regular tetrahedra, it is clear that we can only build 4 irregular octahedra, which we could then try to assemble.
It turns out that between them are formed holes that can however be easily filled with the remaining 4 irregular tetrahedral (3).
The final result is a surprising **truncated cube** (4), lacking the vertices, which, of course, are regular tetrahedra.
Having used all the pieces minus the regular octahedron, we will thus obtain a new proof that:

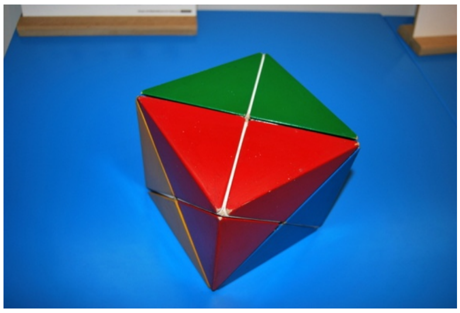
$$Vol_{oct} = 4 \times Vol_{tetr}$$

At the same time, we will have shown that the volume of a cube-octahedron is:

$$Vol_{cube\_oct} = Vol_{cube} - Vol_{oct}$$

More surprising is that the construction of this cube-octahedron is due to two 9-year-old students.

Activity 6
Calculate the volume of the final cube in terms of the volume of a tetrahedron



$Vol_{oub} = Vol_{oct} + 8\times Vol_{tetr1} + 12\times Vol_{tetr2} =$
$= 4\times Vol_{tetr} + 8\times Vol_{tetr1} + 12\times Vol_{tetr2} = 24\times Vol_{tetr}$

Question: Which is the volume of the cube, in terms of the volume of a tetrahedron?

$Vol_{oub} = Vol_{oct} + \frac{8}{4}\times Vol_{tetr1} + \frac{12}{4}\times Vol_{tetr2} =$
$= Vol_{oct} + 2\times Vol_{oct} + 3\times Vol_{oc\ t} = 6\times Vol_{tetr}$

Activity 7: some calculus
Considering the volume of the final cub as the unit, we can express as fractions:
- The volume of the core octahedron;
- The volume of the big tetrahedron;
- The volume of the medium tetrahedron;
- The volume of the initial tetrahedron.

It is easy to calculate that:
The volume of the core octahedron and the volume of the medium tetrahedron are 1/6 of the cube's volume.

The volume of the big tetrahedron is 1/3 of the cube’s volume.
The volume of the tetrahedron is 1/24 of the cube’s volume.

Considering the edge of the final cube as unit, the sides of ALL the pieces are $\frac{2}{2}$ or 1 ,

The area of any of the equilateral triangle is $\frac{\frac{\frac{1}{2}*2}{2}*6}{4}=\frac{3}{8}$

and tetrahedron diagonal is $\frac{3}{3}$ (as you can easily prove observing that any cube diagonal is $\sqrt{3}$ and is formed by three equal segments corresponding to the height of two regular tetrahedral and the distance between octahedron opposite sides).
Now, we can control that our mathematical process is correct calculating once more the volume of a tetrahedron: $\frac{\frac{\frac{1}{3}*\sqrt{3}}{8}*\sqrt{3}}{3}=\frac{1}{24}$